

SPECTRUM OF MIXED BI-UNIFORM HYPERGRAPHS

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ABSTRACT. A mixed hypergraph is a triple $H = (V, \mathcal{C}, \mathcal{D})$, where V is a set of vertices, \mathcal{C} and \mathcal{D} are sets of hyperedges. A vertex-coloring of H is proper if \mathcal{C} -edges are not totally multicolored and \mathcal{D} -edges are not monochromatic. The feasible set $S(H)$ of H is the set of all integers, s , such that H has a proper coloring with s colors.

Bujtás and Tuza [*Graphs and Combinatorics* **24** (2008), 1–12] gave a characterization of feasible sets for mixed hypergraphs with all \mathcal{C} - and \mathcal{D} -edges of the same size r , $r \geq 3$.

In this note, we give a short proof of a complete characterization of all possible feasible sets for mixed hypergraphs with all \mathcal{C} -edges of size ℓ and all \mathcal{D} -edges of size m , where $\ell, m \geq 2$. Moreover, we show that for every sequence $(r(s))_{s=\ell}^n$, $n \geq \ell$, of natural numbers there exists such a hypergraph with exactly $r(s)$ proper colorings using s colors, $s = \ell, \dots, n$, and no proper coloring with more than n colors. Choosing $\ell = m = r$ this answers a question of Bujtás and Tuza, and generalizes their result with a shorter proof.

Keywords: mixed hypergraph, vertex coloring, spectrum, feasible set.

1. INTRODUCTION

A mixed hypergraph is a triple $H = (V, \mathcal{C}, \mathcal{D})$, where V is a set of vertices, \mathcal{C} is a set of subsets of vertices, called \mathcal{C} -edges, and \mathcal{D} is a set of subsets of vertices, called \mathcal{D} -edges. A coloring $c : V \rightarrow \mathbb{N}$ of H is proper if each \mathcal{C} -edge has at least two vertices of the same color, and each \mathcal{D} -edge has at least two vertices of distinct colors, i.e., \mathcal{C} -edges are not rainbow and \mathcal{D} -edges are not monochromatic. An s -coloring of H is a coloring using exactly s colors. The notion of mixed hypergraphs was introduced by Voloshin [7]. Their importance was emphasized by Král, who showed connections between feasible coloring of mixed hypergraphs and other classical coloring problems such as list-colorings of graphs, see [5]. One of the important parameters of a mixed hypergraph H , introduced in [7], is its *chromatic spectrum*, that is the sequence $(r_s(H))_{s=1}^n$, where $n = |V(H)|$, with

$$r_s(H) = \# \text{ non-isomorphic proper } s\text{-colorings of } H$$

for $s = 1, \dots, n$. Here, two colorings are *isomorphic* if they are the same up to relabeling the colors. The set $\{s \in \mathbb{N} \mid r_s(H) \neq 0\}$ is called the *feasible set* of H , i.e., it is the set of natural numbers

$$S(H) = \{s : \text{there is a proper } s\text{-coloring of } H\}.$$

Král proved that for any finite $S \subset \mathbb{N}$, $1 \notin S$, and any function $r : S \rightarrow \mathbb{N} \setminus \{0\}$ there is a mixed hypergraph H with $S(H) = S$ and $r_s(H) = r(s)$ for every $s \in S$. This extended a previous result by Jiang *et al.* [3]. Bujtás and Tuza [1] described the feasible sets of *uniform mixed hypergraphs*, i.e., where \mathcal{C} - and \mathcal{D} -edges have the same cardinality. To state the results, we always shall assume that the mixed hypergraphs are non-empty, i.e., $\mathcal{C} \cup \mathcal{D} \neq \emptyset$. We denote the interval of integers $\{i, i+1, \dots, j\}$, $i \leq j$, as $[i, j]$.

Theorem 1 (Bujtás, Tuza [1]).

Let $r \geq 3$ be an integer, and S be a finite set of natural numbers. There is a non-empty r -uniform mixed hypergraph $H = (V, \mathcal{C}, \mathcal{D})$ with $S(H) = S$ if and only if

- (A1) $S = [1, x]$, for some $x \geq r - 1$, or
- (A2) $\min(S) \geq r$, or
- (A3) $\min(S) = a \in [2, r - 1]$, and $[a, r - 1] \subseteq S$.

Zhao *et al.* [8] proved that if $r = 3$ and S is of type (A2) or (A3), then for every function $r : S \rightarrow \mathbb{N} \setminus 0$ there is a 3-uniform mixed hypergraph H with $S(H) = S$ and $r_s(H) = r(s)$ for every $s \in S$.

A mixed hypergraph $H = (V, \mathcal{C}, \mathcal{D})$ is (ℓ, m) -uniform if every C -edge has size ℓ and every D -edge has size m , i.e., $\mathcal{C} \subseteq \binom{V}{\ell}$ and $\mathcal{D} \subseteq \binom{V}{m}$. Here, we provide a complete characterization of feasible sets for (ℓ, m) -uniform mixed hypergraphs for any $\ell, m \geq 2$. We also characterize the spectra of such hypergraphs in the range where at least ℓ colors are used.

Theorem 2. Let $\ell, m \geq 2$ be integers, and S be a finite set of natural numbers. There is a non-empty (ℓ, m) -uniform mixed hypergraph $H = (V, \mathcal{C}, \mathcal{D})$ with $S(H) = S$ if and only if

- (B1) $S = [1, x]$ for some $x \geq \ell - 1$, or
- (B2) $\ell = 2$ and S is an arbitrary interval, or
- (B3) $\ell \geq 3$ and $\min(S) \geq \ell$, or
- (B4) $\ell \geq 3$ and $\min(S) = a \in [2, \ell - 1]$ and $[a, \ell - 1] \subseteq S$.

Moreover, if $r : S \rightarrow \mathbb{N} \setminus 0$ is a function, and S is of type (B3) or (B4), then there is such a hypergraph H with $S = S(H)$ and $r_s(H) = r(s)$ for every $s \in S$, $s \geq \ell$.

Note that in case (B3) and (B4) the constructed hypergraph H does not necessarily satisfy $r_s(H) = r(s)$ when $s < \ell$. In fact, when $s < \ell$ one *can not* prescribe the value of $r_s(H)$, as for example there is no (ℓ, m) -uniform mixed hypergraph H with $r_s(H) \geq 1$ and $r_{s+1}(H) = 1$ for $s + 1 < \ell \leq |V(H)|$. Indeed, taking any proper s -coloring and partitioning its largest color class in different ways into two non-empty parts, provides at least two different proper $(s + 1)$ -colorings.

We shall prove Theorem 2 in Section 2 below by following the same underlying approach as Bujtás and Tuza [1], Zhao *et al.* [8], as well as Cherubini [2]. First, we provide an easy argument for the necessity of (B1)–(B4). Then we construct, for any $m \geq 2$, $\ell \geq 3$ and $1 < a < \ell \leq b$, an (ℓ, m) -uniform hypergraph $H = H(\ell, m, a, b)$ with $S(H) = [a, \ell - 1] \cup \{b\}$ and $r_b(H) = 1$. Finally, we construct a more general hypergraph for any given feasible set of type (B3) or (B4) and any given chromatic spectrum whose “building blocks” are the hypergraphs constructed before.

2. PROOFS

We start with some easy observations.

Lemma 3. Let $H = (V, \mathcal{C}, \mathcal{D})$ be an (ℓ, m) -uniform mixed hypergraph with at least $\ell - 1$ vertices.

- If $1 \in S(H)$, then $S(H) = [1, x]$, $x \geq \ell - 1$.
- If $\ell = 2$, then $S(H)$ is an interval.
- If $a \in S(H)$ for $2 \leq a \leq \ell - 1$, then $[a, \ell - 1] \subseteq S(H)$.

Proof. If $1 \in S(H)$, then $\mathcal{D} = \emptyset$. Consider any proper x -coloring of H with $x = \max(S(H))$. Observe that merging two color classes results again in a proper coloring

of H , as $\mathcal{D} = \emptyset$. Thus $S(H) = [1, x]$. On the other hand, any coloring with $\ell - 1$ colors is a proper coloring of H . Thus $x \geq \ell - 1$.

If $\ell = 2$, then in every proper coloring of H all vertices in the same connected component of the graph $G = (V, \mathcal{C})$ receive the same color. Hence we have $S(H) = S(H')$, where H' is the hypergraph that arises from H by contracting all vertices in the same connected component of G into a single vertex and removing all C -edges. Since H' has only D -edges, $S(H')$ is an interval. Note that H' , and hence H , is uncolorable if H' contains a D -edge of size 1.

For $a \in S(H)$, $2 \leq a \leq \ell - 1$, consider any proper a -coloring c of H . If $a = \ell - 1$ there is nothing to show. Otherwise, if we subdivide color classes in c into new color classes so that the resulting number of colors is at most $\ell - 1$, then the coloring is still proper. Indeed, each C -edge has ℓ vertices, thus at least two vertices of the same color. Each D -edge still uses distinct colors on its vertices. Such a subdivision exists whenever H has at least $\ell - 1$ vertices. \square

2.1. Construction of a hypergraph $H = H(\ell, m, a, b)$ for $1 < a < \ell \leq b$.

Let $V = V(H) = X \times Y \times Z$, where $X = [a]$, $Y = [b]$ and $Z = [m]$. We shall also refer to $\{x\} \times Y \times Z$ as a *row*, and $X \times \{y\} \times Z$ as a *column* of H . The set \mathcal{C} of C -edges consists of all ℓ -element sets containing at least 2 vertices from the same column. The set \mathcal{D} of D -edges consists of all m -element sets that are *not* completely contained in any row and *not* completely contained in any column.

We say that a coloring is a *row-coloring* of H if it assigns the same color to all vertices in each row such that any two vertices from distinct rows get distinct colors. We say that a coloring is a *column-coloring* of H if it assigns the same color to all vertices in each column such that any two vertices from distinct columns get distinct colors. Note that both row- and column-coloring are proper colorings of H . Indeed, every C -edge contains two vertices from the same column by definition and two vertices from the same row since $a < \ell$. Every D -edge contains two vertices from distinct rows and two vertices from distinct columns by definition. We refer to Figure 1 for an illustration.

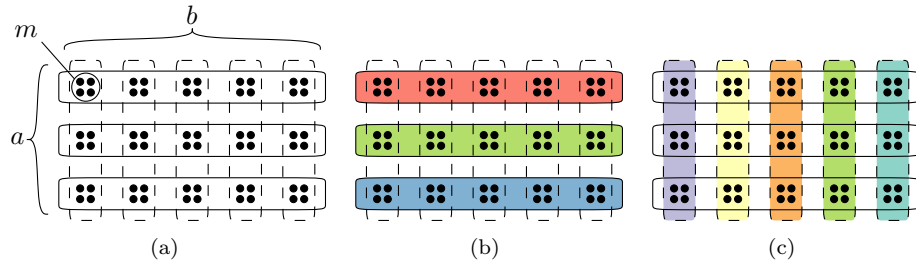


FIGURE 1. Hypergraph $H(\ell, m, a, b)$ with $a = 3$, $b = 5$ and $m = 4$. Rows are highlighted in solid and columns in dashed. A row-coloring and a column-coloring are indicated in (b) and (c), respectively.

Theorem 4. *If $H = H(\ell, m, a, b)$, $1 < a < \ell \leq b$, then $S(H) = [a, \ell - 1] \cup \{b\}$. Moreover, every proper a -coloring of H is a row-coloring and every proper b -coloring of H is a column-coloring.*

Proof. First, we claim that $(V(H), \mathcal{D})$ is an a -chromatic hypergraph, and hence each proper coloring of H uses at least a colors. Indeed, if H is colored with less than a colors, then the biggest color class contains more than $b \cdot m$ vertices, and thus contains two vertices from distinct rows and columns. These two vertices together with some other $(m - 2)$ vertices from that color class gives a monochromatic D -edge, a contradiction.

Now observe that a row-coloring of H uses exactly a colors, which with Lemma 3 implies that $[a, \ell - 1] \subseteq S(H)$. Moreover, we claim that every proper coloring that uses exactly a colors is a row-coloring. Indeed, we can argue by induction on a , with the case $a = 1$ being immediate. For $a \geq 2$, consider a proper a -coloring of H and a largest color class V' . Then $|V'| \geq bm$. If V' contains two vertices from at least two different rows and different columns, these two vertices together with some other $(m - 2)$ vertices from V' give a monochromatic D -edge, a contradiction. Thus, V' is contained in one fixed row and since $|V'| \geq bm$, it is actually equal to that row. Removing this row gives a proper $(a - 1)$ -coloring of $a - 1$ rows, to which we can apply induction.

A column-coloring of H is a proper coloring using exactly b colors. In general, if each column of H is monochromatic, then the columns must use distinct colors because of the D -edges. So, if all columns are monochromatic, then the coloring is a column-coloring and it uses exactly b colors. If some column is not monochromatic and the total number of colors is at least ℓ , then taking two vertices v, v' of distinct colors from that column and $\ell - 2$ vertices of distinct colors different from the colors of v and v' gives a rainbow C -edge, a contradiction. This concludes the proof. \square

2.2. Construction of a hypergraph $H = H(\ell, m, a, b_1, b_2, \dots, b_q)$, $1 < a < \ell \leq b_1 \leq b_2 \leq \dots \leq b_q$, and a coloring $\mathbf{c}_i(H)$.

First consider the hypergraphs H_1, H_2, \dots, H_q , where $H_i = H(\ell, m, a, b_i)$, $i = 1, \dots, q$, on pairwise disjoint sets of vertices. If $q = 1$, we define $H = H_1$. Otherwise, the hypergraph H is the union of H_i 's, $i = 1, \dots, q$, and an additional set of C -edges, denoted \mathcal{C}_{add} , where $\mathcal{C}_{add} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$, is defined as below. Let the vertex set of H be V . A subset of a row (respectively column) of some H_i , $1 \leq i \leq q$, is called a *transversal* of that row (respectively column) if it contains at most one vertex from every column (respectively row) of H_i . We say that a triple of vertices $v, v', v'' \in V$ is *special* in H_i , $1 \leq i \leq q$, if v and v' are in the same row, and v and v'' are in different rows but the same column of H_i . We define $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ implicitly by saying that for any $V' \in \binom{V}{\ell}$ we have

- $V' \in \mathcal{C}_1$ if V' contains a special triple of H_i for some $1 \leq i \leq q$,
- $V' \in \mathcal{C}_2$ if $V' = V_1 \dot{\cup} V_2$, V_1 is a transversal of size a of the 1^{st} column of H_i and $V_2 \subseteq V(H_{i+1}) \cup \dots \cup V(H_q)$, for some $1 \leq i \leq q - 1$,
- $V' \in \mathcal{C}_3$ if $V' = V_1 \dot{\cup} V_2$, V_1 is a transversal of size $\ell - 2$ of the 1^{st} row of H_i and V_2 consists of two vertices of the 1^{st} row of H_{i+1} , one of which has a column index that coincides with the column index of some vertex in V_1 , for some $1 \leq i \leq q - 1$, and
- $V' \in \mathcal{C}_4$ if $V' = V_1 \dot{\cup} V_2$, V_1 is a transversal of size $\ell - 2$ of the k^{th} row of H_i containing a vertex from the k^{th} column and V_2 consists of two vertices of the 1^{st} column of H_j , one of which from the k^{th} row of H_j , for some $1 \leq i \leq q - 1$, $1 \leq k \leq a$, $i < j \leq q$.

Now, we define a coloring $\mathbf{c}_i(H)$, $i = 0, 1, \dots, q$. If $i < q$ color H_{i+1}, \dots, H_q with the row-colorings using color k on the k^{th} row, $k = 1, \dots, a$, and color the remaining hypergraphs (if $i \geq 1$) H_1, \dots, H_i with the column-colorings in which color k is used on the k^{th} column in each of these hypergraphs. See Figure 3.

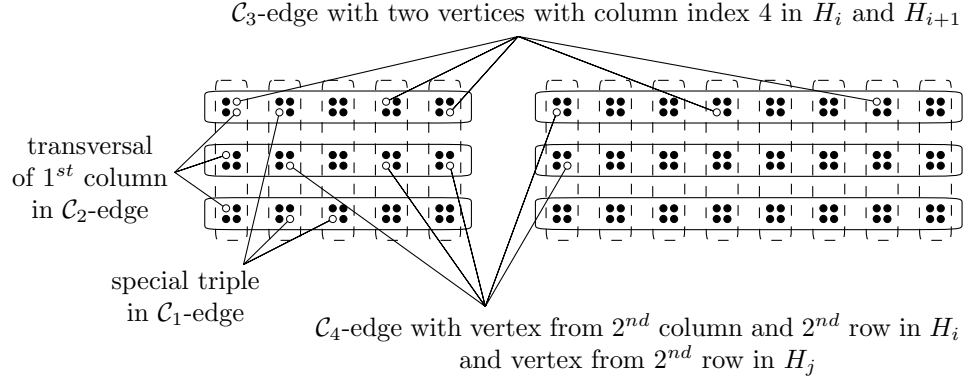


FIGURE 2. Illustration of $H(\ell, m, a, b_1, b_2)$ and one \mathcal{C}_1 -, \mathcal{C}_2 -, \mathcal{C}_3 - and \mathcal{C}_4 -edge, where $\ell = 5$, $m = 4$, $a = 3$, $b_1 = 5$ and $b_2 = 8$.

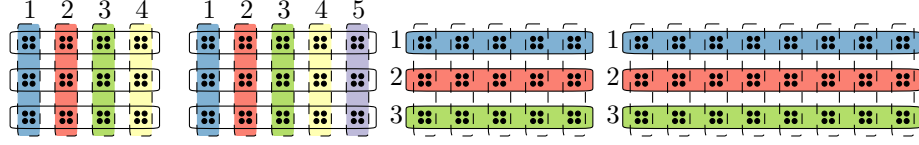


FIGURE 3. The coloring $\mathbf{c}_2(H)$ of $H(\ell, m, a, b_1, \dots, b_q)$. Here $a = 3$, $\ell = 4$, $m = 6$, $q = 4$ and $(b_1, \dots, b_q) = (4, 5, 5, 8)$. Indices indicate the colors of corresponding rows or columns.

Theorem 5.

For all $m \geq 2$ and $\ell \geq 3$ the (ℓ, m) -uniform hypergraph $H = H(\ell, m, a, b_1, b_2, \dots, b_q)$, $1 < a < \ell \leq b_1 \leq b_2 \leq \dots \leq b_q$, has the feasible set $S(H) = [a, \ell - 1] \cup \{b_1, b_2, \dots, b_q\}$.

Moreover, if $s = b_i$ for exactly $r(s)$ values of $i \in \{1, \dots, q\}$, then $r_s(H) = r(s)$, $s \geq \ell$.

Proof. Let c be a proper coloring of H . We shall denote by $c(H_i)$ the set of colors used on H_i , and by $c(H)$ the set of colors used on H . We shall show the following.

Claim 1. One of the following must happen up-to relabeling of the colors:

- $|c(H)| \leq \ell - 1$, or
- there is an index i such that $c = \mathbf{c}_i(H)$.

We prove Claim 1 through a series of claims, one for each type of \mathcal{C} -edges in \mathcal{C}_{add} .

Claim 2. Either $a < |c(H)| \leq \ell - 1$ or each H_i is row- or column-colored.

Assume that for some $i \in \{1, \dots, q\}$, H_i is neither row- nor column-colored, then we have to show that $|c(H)| \leq \ell - 1$. From Theorem 4 it follows immediately that $a < |c(H_i)| \leq \ell - 1$. First, we shall show that there is a rainbow special triple in H_i . Indeed, since $|c(H_i)| > a$ and H_i has a rows, there are two rows R_1, R_2 with at least 3 colors on them. If each column in $R_1 \cup R_2$ were monochromatic, then since $|c(H_i)| \leq \ell - 1$, there would be two monochromatic columns there in the same color, and thus a monochromatic D -edge. Thus, there are vertices $v_1 \in R_1$, $v_2 \in R_2$, v_1, v_2 from the same column, $c(v_1) \neq c(v_2)$. Then a vertex v of a third color in $R_1 \cup R_2$ forms a special rainbow triple together with v_1 and v_2 . If $|c(H)| \geq \ell$ then $\{v, v_1, v_2\}$ together with $\ell - 3$ vertices of different colors not appearing on this triple form a rainbow \mathcal{C}_1 -edge, a contradiction. Thus $|c(H)| \leq \ell - 1$.

From now on, we assume that each H_i is row- or column-colored.

Claim 3. *If H_i is row-colored, then $|\bigcup_{j=i}^q c(H_j)| \leq \ell - 1$ and each of H_{i+1}, \dots, H_q is also row-colored.*

Assume that H_i is row-colored and $|\bigcup_{j=i}^q c(H_j)| \geq \ell$. Then any transversal V_1 of the 1st column of H_i together with $\ell - a$ vertices of $\bigcup_{j=i+1}^q H_j$ having distinct colors different from the colors in V_1 forms a rainbow \mathcal{C}_2 -edge, a contradiction. Since a column-coloring uses at least ℓ colors, each of H_{i+1}, \dots, H_q is not column-colored, thus it is row-colored.

Claim 4. *If H_i and H_{i+1} are column-colored, $1 \leq i < q$, then the colors of the k^{th} column of H_i and H_{i+1} coincide, $1 \leq k \leq b_i$.*

Assume not. Let $v, v' \in V(H_{i+1})$ be from the 1st row, and the k^{th} , k'^{th} column, respectively, $k \neq k'$. Let V_1 be a transversal of the 1st row of H_i of size $\ell - 2$, containing a vertex v'' from the k^{th} column and using none of $c(v), c(v')$. Such V_1 exists, since $b_i \geq \ell$ and $c(v'') \neq c(v)$. Then $V_1 \dot{\cup} \{v, v'\}$ is a rainbow \mathcal{C}_3 -edge, a contradiction.

Claim 5. *If H_i is column-colored and H_j is row-colored, $1 \leq i < j \leq q$, then the colors of the k^{th} column of H_i and the k^{th} row of H_j coincide, $1 \leq k \leq a$.*

Assume not. Let $v, v' \in V(H_j)$ be from the 1st column and the k^{th} , k'^{th} rows, respectively, $k \neq k'$. Let V_1 be a transversal of size $\ell - 2$ of the k^{th} row of H_i containing a vertex from the k^{th} column, and using none of $c(v), c(v')$. Since $b_i \geq \ell$, such a transversal exists. Then $V_1 \dot{\cup} \{v, v'\}$ is a rainbow \mathcal{C}_4 -edge, a contradiction.

To confirm Claim 1, observe that Claim 2 implies that either $|c(H)| \leq \ell - 1$ or each H_i is row- or column-colored. Claims 3, 4, 5 confirm that if each H_i is row- or column-colored, then $c = \mathbf{c}_i(H)$, for some index i . To finish the proof we need one last claim.

Claim 6. *The coloring $\mathbf{c}_0(H)$ is a proper a -coloring and the coloring $\mathbf{c}_i(H)$ is a proper b_i -coloring, $i = 1, \dots, q$.*

Indeed, every special triple contains two vertices from the same column and two vertices from the same row, and hence no \mathcal{C}_1 -edge is rainbow. No \mathcal{C}_2 -edge is rainbow, since it either contains two vertices from the same column in a column-colored H_i , or is completely contained in a subgraph that is $(\ell - 1)$ -colored. No \mathcal{C}_3 -edge is rainbow, since it contains two vertices from H_i and H_{i+1} , $1 \leq i < q$, with the same column index and two vertices from H_{i+1} from the same row. No \mathcal{C}_4 -edge is rainbow, since it contains a vertex v from H_i and a vertex v' from H_j , $1 \leq i < j \leq q$, where row index and column index of v and row index of v' coincide, and two vertices from H_j from the same column.

Continuing with the proof of Theorem 5, recall that $|c(H)| \geq \chi((V, \mathcal{D})) \geq a$. Claim 1 implies that $S(H) \subseteq [a, \ell - 1] \cup \{b_1, b_2, \dots, b_q\}$. Indeed, either the number of colors is at most $\ell - 1$ or $c(H) = \mathbf{c}_i(H)$, $|\mathbf{c}_i(H)| = b_i$, $i = 1, \dots, q$. This also implies that $r_s(H) \leq r(s)$ for every $s \in \{b_1, \dots, b_q\}$.

It remains to show that $S(H) \supseteq [a, \ell - 1] \cup \{b_1, b_2, \dots, b_q\}$ and that $r_s(H) \geq r(s)$ for every $s \in \{b_1, \dots, b_q\}$. When $s = a$, consider the coloring $\mathbf{c}_0(H)$, which is a proper a -coloring by Claim 6. When $s \in [a, \ell - 1]$, Lemma 3 implies the existence of a proper s -coloring of H . When $s \in \{b_1, \dots, b_q\}$, consider a maximal block of repeated s , i.e., take largest j and smallest i such that $b_i = b_{i+1} = \dots = b_j = s$. So, $r(s) = j - i + 1$. Consider the colorings $\mathbf{c}_i(H), \dots, \mathbf{c}_j(H)$. Claim 6 implies that these are $r(s)$ pairwise non-isomorphic proper s -colorings of H . Thus, $r_s(H) \geq r(s)$ for every $s \in \{b_1, \dots, b_q\}$ and $S(H) \supseteq [a, \ell - 1] \cup \{b_1, b_2, \dots, b_q\}$. \square

2.3. Proof of Theorem 2. Having Lemma 3, Theorem 4 and Theorem 5, we can finally prove our main result.

Proof. Lemma 3 gives the necessity of (B1)–(B4). Note that if $|V(H)| \leq \ell - 1$, then H contains only D -edges and then $S(H)$ is an interval. It remains to construct a hypergraph with the desired properties for any given S of type (B1), (B2), (B3), or (B4).

Let S be of type (B1), i.e., $S = [1, x]$, $x \geq \ell - 1$. Consider a hypergraph $H = (V, \mathcal{C}, \mathcal{D})$ with $\mathcal{D} = \emptyset$, $|V| = x + 1$, $v, v' \in V$, and \mathcal{C} consisting of all ℓ -subsets of V containing v and v' . In every s -coloring with $s > x$ the vertex set is totally multicolored, so no C -edge is properly colored. On the other hand, we obtain a proper s -coloring of H for $s \leq x$ by distributing s colors arbitrarily, so that v and v' receive the same color.

Let $\ell = 2$ and S be of type (B2), i.e., $S = [a, b]$. Define a $(2, m)$ -uniform hypergraph H as follows. The vertex set $V(H)$ consists of b disjoint sets V_1, \dots, V_b of m vertices each, and \mathcal{C} is the set of all pairs of vertices from the same V_i , $i = 1, \dots, b$. For $a > 1$, we define \mathcal{D} to be the set of all m -subsets of $V(H)$ containing vertices from exactly two V_i 's with $i \in \{1, \dots, a\}$, and no vertex from $V_{a+1} \cup \dots \cup V_b$. For $a = 1$, we define $\mathcal{D} = \emptyset$. Then it is easy to verify that $S(H) = [a, b]$.

Let $\ell \geq 3$ and S be of type (B3), i.e., $\min(S) \geq \ell$ and let $r : S \rightarrow \mathbb{N} \setminus 0$ be a given function. Let $b_1 \leq b_2 \leq \dots \leq b_q$ be nonnegative integers such that $\{b_1, \dots, b_q\} = S$ and each $b \in S$ be repeated in the list exactly $r(b)$ times. For example, when $S = \{3, 5\}$, $r(3) = 2, r(5) = 1$, then $(b_1, b_2, b_3) = (3, 3, 5)$.

Let H be a graph formed by adding an extra D -edge to the part H_1 of $H' = H(\ell, m, \ell - 1, b_1, \dots, b_q)$, where this edge is contained in a row of H_1 , but not in any of its columns. By Theorem 5, H' has feasible set $\{\ell - 1\} \cup S$ and there are exactly $r(s)$ proper s -colorings of H' for every $s \in S$. By Theorem 4, every proper $(\ell - 1)$ -coloring of H' is a row-coloring, thus H is no longer $(\ell - 1)$ -colorable because of the extra D -edge. On the other hand, the b_i -coloring $\mathbf{c}_i(H - \{D''\})$ is still a proper coloring of H , $i = 1, \dots, q$, which implies that $S(H) = S$ and $r_s(H) = r(s)$ for every $s \in S$.

Finally consider the case when $\ell \geq 3$ and S is of type (B4), i.e., $S = [a, \ell - 1] \cup S'$ for some $a \in [2, \ell - 1]$ and $S' = \emptyset$ or $\min(S') \geq \ell$. Let $r : S \rightarrow \mathbb{N} \setminus 0$ be a given function. For $S' \neq \emptyset$, consider the hypergraph $H = H(\ell, m, a, b_1, \dots, b_q)$, where every $s \in S'$ appears exactly $r(s)$ times in the list (b_1, \dots, b_q) . By Theorem 5, H has the desired properties. For $S' = \emptyset$, let H be obtained from $H' = H(\ell, m, a, b)$ with some b , $b \geq \ell$, by adding an extra D -edge that is contained in a column of H' , but not in a row of H' . Since by Theorem 4 every proper b -coloring of H' is a column-coloring, H is no longer b -colorable. On the other hand, the row-coloring using color k on k^{th} row, $k = 1, \dots, a$, of H' is a proper a -coloring of H , which implies that $S(H) = [a, \ell - 1]$. \square

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